Global structure of the Zipoy-Voorhees-Weyl spacetime and the $\delta = 2$ Tomimatsu-Sato spacetime

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We investigate the structure of the ZVW (Zipoy-Voorhees-Weyl) spacetime, which is a Weyl solution described by the Zipoy-Voorhees metric, and the $\delta = 2$ Tomimatsu-Sato spacetime. We show that the singularity of the ZVW spacetime, which is represented by a segment $\rho = 0, -\sigma < z < \sigma$ in the Weyl coordinates, is geometrically point-like for $\delta < 0$, string-like for $0 < \delta < 1$ and ring-like for $\delta > 1$. These singularities are always naked and have positive Komar masses for $\delta > 0$. Thus, they provide a non-trivial example of naked singularities with positive mass. We further show that the ZVW spacetime has a degenerate Killing horizon with a ring singularity at the equatorial plane for $\delta = 2, 3$ and $\delta \geq 4$. We also show that the $\delta = 2$ Tomimatsu-Sato spacetime has a degenerate horizon with two components, in contrast to the general belief that the Tomimatsu-Sato solutions with even δ do not have horizons.

I. INTRODUCTION

Almost thirty years have passed since Tomimatsu and Sato[1] found a series of exact solutions, the so-called TS solutions, to the vacuum Ernst equation, which can be regarded as representing quadrupole deformations of the Kerr solution. However, the global structure of these solutions has not been studied well, or at least, its details are not known widely, though some basic features such as the existence of naked ring singularities and causality violating regions are well known[2–4]. Furthermore, some conflicting or wrong statements on their global and singularity structures are sometimes found in the literature for the case the deformation parameter δ is even. For example, Gibbons and Russell-Clark[4] claimed that the TS (Tomimatsu-Sato) solution with $\delta = 2$, which is referred to as TS2 in the present paper, does not have a horizon, by showing that the spacetime is inextendible across the segment I: $\rho = 0, -\sigma < z < \sigma$ in the Weyl coordinates, which corresponds to a horizon for odd δ[3]. Although they noticed that TS2 has quasi-regular directional singularities at $P_\pm$: $(\rho, z) = (0, \pm\sigma)$ in the Weyl coordinates, and that the spacetime has two-dimensional extensions along the z-axis across these points, they failed to recognize that these points are horizons. Subsequently, Ernst[5] pointed out that these points are hypersurfaces, and investigated the behavior of geodesics crossing them by introducing a polar coordinate system around these points. However, he gave the incorrect conclusion that the corresponding surfaces are time-like and a particle which crossed one of the hypersurfaces can come out again through the same hypersurface. Later, Papadopoulos et al[6] recognized that the two points $P_\pm$ of the ZVW spacetime with $\delta \geq 2$, which is a Weyl solution described by the Zipoy-Voorhees metric[7–9] and a static limit of the TS solution for integer values of δ, are also hypersurfaces and tried to construct a coordinate system providing a smooth extension of the spacetime across these hypersurfaces. They found a two-dimensional extension along the z-axis, but were not able to find a coordinate system giving a four-dimensional extension.

One main reason for the lack of detailed studies may be the fact that the TS solutions and

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the Weyl solutions have pathological features including the existence of naked singularities. If the cosmic censorship hypothesis[10] were correct, such solutions would not describe the real world. However, the cosmic censorship is still a hypothesis, and there are now lots of counter examples[11–15]. Of course, it is probable that the cosmic censorship hypothesis turns out to be correct in practice, since most of the analytic counter examples found so far are spherically symmetric[16]. Hence, in order to clarify whether the cosmic censorship is correct or not, it is inevitable to study non-spherically symmetric systems[17]. For that purpose, knowledge on possible structures of spacetimes with naked singularities will give useful information. Such knowledge will be also important if the cosmic censorship turns out to be incorrect. From this point of view, in the present paper, we investigate in detail the global structure of the ZVW and TS spacetimes, with focus on singularities and horizons. Although we do not restrict the deformation parameter δ for the ZVW solution, we only consider the $\delta = 2$ case for the TS solution, because it is hard to analyse the TS solutions with $\delta \geq 3$, even by using symbolic computations by computers.

We will reveal lots of new features of the spacetimes described by these solutions concerning the singularity and causal structure. In particular, we will prove that the ZVW spacetimes with $\delta = 2, 3$ and $\delta \geq 4$ have a degenerate Killing horizon at $\mathrm{P}_\pm$, as anticipated by Papadopoulos et al, and show that the horizon has a ring singularity at the equator. We further show that at least for $\delta = 2$, the Tomimatsu-Sato spacetime has degenerate Killing horizons at $\mathrm{P}_\pm$, each of which is a sphere with conic singularity. We will also point out that the ZVW solution with $\delta > 0$ ($\delta \neq 1$) provides a rare example of non-trivial naked singularity with positive gravitational mass.

The paper is organized as follows. In section II, we first study in detail the structure of the singularities of the ZVW spacetimes, and then show that for $\delta = 2, 3$ and $\delta \geq 4$, the directional quasi-regular singularities at $\mathrm{P}_\pm$ are regular parts of a degenerate Killing horizon, by giving explicit regular extensions of the spacetime across these points. In section III, we extend the analysis to TS2. After a brief review of the basic features of this spacetime, we prove that the directional quasi-regular singularities at $\mathrm{P}_\pm$ are degenerate Killing horizons. We also clarify the structure of singularity at the segment I on the z-axis. Section IV is devoted to summary and discussion. All symbolic computations in the present paper were done by Maple V.

II. ZIPOY-VOORHEES-WEYL SPACETIME

The ZVW spacetimes are a two-parameter family of static, axisymmetric and asymptotically flat vacuum solutions to the Einstein equations described by the Zipoy-Voorhees metric[7–9], which include the flat and Schwarzschild solutions. In the canonical form for the static axisymmetric metric

$$ds^2 = -f dt^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \tag{1}$$

they are expressed as[8]

$$f = \left(\frac{x-1}{x+1}\right)^{\delta}, \quad e^{2\gamma} = \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}. \tag{2}$$

Here, x and y are the prolate spheroidal coordinates related to the Weyl coordinates (ρ, z) by

$$\rho = \sigma\sqrt{(x^2-1)(1-y^2)}, \quad z = \sigma x y. \tag{3}$$

In this prolate spheroidal coordinate system (t, ϕ, x, y), the ZVWmetric is written as

$$ds^2 = -f dt^2 + R^2 d\phi^2 + \Sigma^2 \left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right), \tag{4}$$

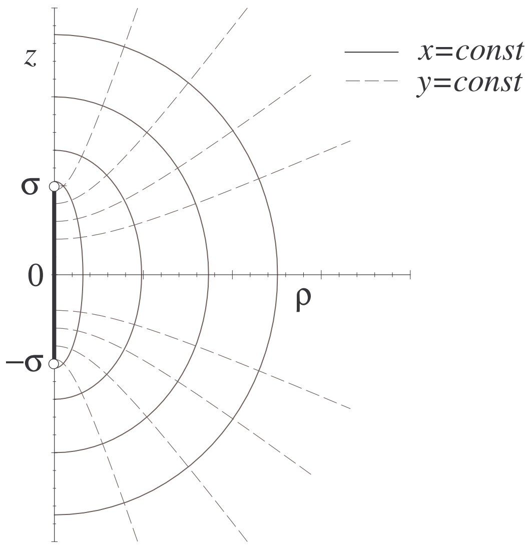

FIG. 1: The structure of the ZVW spacetime. The thick solid line and the two open circles denote the curvature singularity and the directional singularities, respectively. The x =const and y =const curves are also shown.

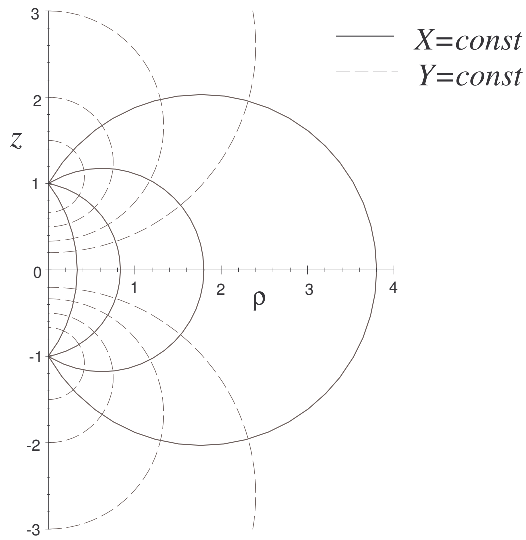

FIG. 2: The relation between the (ρ, z) and the (X, Y) coordinate systems in the units $\sigma = 1$. The solid and the dotted curves are X =const and Y =const curves, respectively.

where

$$R^2 = \sigma^2 \left(\frac{x+1}{x-1}\right)^{\delta-1} (x+1)^2(1-y^2), \tag{5}$$

$$\Sigma^2 = \sigma^2 \frac{(x+1)^{\delta^2+\delta}}{(x-1)^{-\delta^2+\delta}}(x^2-y^2)^{1-\delta^2}. \tag{6}$$

Here, note that the metric is invariant under the transformation $\delta \to -\delta, x \to -x$. Further, the metric has a curvature singularity at some value of x, except for the case $\delta = 0$, as we see below. Hence, we can assume that the asymptotically flat region corresponds to the range $x > 1$ and $-1 \le y \le 1$. In this asymptotic flat region, for the spherical coordinates (r, θ, ϕ) defined by $\sigma x = r - \delta\sigma + \mathrm{O}(1/r)$ and $y = \cos\theta$, the metric components in (4) are expanded as

$$f = 1 - \frac{2\sigma\delta}{r} + \mathrm{O}\left(\frac{1}{r^3}\right), \tag{7a}$$

$$R^2/\sin^2\theta = r^2 + 2\sigma\delta r + (4\delta^2-1)\sigma^2 + \mathrm{O}\left(\frac{1}{r}\right), \tag{7b}$$

$$\Sigma^2 = r^2 + 2\sigma\delta r + [4\delta^2 - 1 + (1-\delta^2)\sin^2\theta]\sigma^2 + \mathrm{O}\left(\frac{1}{r}\right). \tag{7c}$$

From these expressions, we see that the gravitational mass of the system is represented as $M = \delta\sigma$, and that the parameter δ taking a real number describes a quadrupole deformation of the spacetime[7, 8]: each f =constant surface has a prolate shape for $|\delta| < 1$ and an oblate shape for $|\delta| > 1$. In particular, $\delta = 0$ corresponds to the flat metric, and $\delta = \pm 1$ to the Schwarzschild metric with positive or negative mass.

A. Singularity structure

From (4), it is clear that the metric can have curvature singularity only at $x = \pm 1$ and $x = \pm y$. In this section, we first confirm that there is actually singularity at $x = \pm 1$ for $\delta \neq 0, 1$ and investigate the structure of the singularity there. In the present paper, we focus on singularities accessible from the asymptotically flat region $x > 1, -1 \leq y \leq 1$, and do not consider possible singularities at $x = \pm y$, which are separated from the asymptotically flat region by naked singularities or horizons.

For a static and axisymmetric metric, $\Psi_1 = \Psi_3 = 0$ and all scalar polynomials of the Weyl curvature can be written as polynomials of Ψ_2 and $\Psi_0\Psi_4$, where $\Psi_0 \sim \Psi_4$ are the Newman-Penrose components of the Weyl curvature with respect to the complex null tetrad $(k, l, m, \bar{m})$ defined by

$$k_\mu dx^\mu = \frac{\sqrt{f}dt + Rd\phi}{\sqrt{2}}, \quad l_\mu dx^\mu = \frac{\sqrt{f}dt - Rd\phi}{\sqrt{2}}, \quad m_\mu dx^\mu = \frac{e^\gamma(d\rho - idz)}{\sqrt{2f}}. \tag{8}$$

For the metric (4), these curvature invariants are given by

$$\Psi_2 = \frac{\delta(x-\delta)(x^2-y^2)^{\delta^2-1}}{2(x-1)^{\delta^2-\delta+1}(x+1)^{\delta^2+\delta+1}}, \tag{9a}$$

$$\begin{aligned}\Psi_0\Psi_4 = & \frac{\delta^2(x^2-y^2)^{2\delta^2-3}}{4[(x-1)^{\delta^2-\delta+1}(x+1)^{\delta^2+\delta+1}]^2} \times \Big[9(x-1)^2(x^2-y^2) \\ & +(\delta-1)\big\{[12(\delta+1)x^2 - 6(2\delta^2+2\delta+3)x + (\delta+1)(4\delta^2+5)](1-y^2) \\ & -9(2x-\delta-1)(x^2-1)\big\}\Big].\end{aligned} \tag{9b}$$

From these expressions, we immediately see that the curvature invariants diverge on the open segment I: $x = 1, -1 < y < 1$, if $\delta \neq 0, 1$. This segment corresponds to the segment $\rho = 0, -\sigma < z < \sigma$ in the Weyl coordinates, which is depicted by a thick line in Fig.1. It is also easy to see that one can reach any point on this segment by a curve with a finite proper length in the $x - y$ plane. As is expected from this, this segment is a naked singularity. In fact, we can show that in the case $L = 0$ and $\epsilon = 0$ (null geodesics), the geodesic equation (A1d) has a regular solution expressed by a power series of $x - 1$ as

$$y = y_0 + \frac{\delta+1}{\delta-1}y_0(x-1) + \cdots, \tag{10}$$

for any y_0 in the open interval $(-1, 1)$, provided that $\delta \neq 1$. From (A1c), we see that the null geodesic represented by this solution reaches the point $(x, y) = (1, y_0)$ in a finite affine parameter. This implies that each point in the open segment $\rho = 0, -\sigma < z < \sigma$ is a naked curvature singularity in the standard sense. By looking for a solution of the type $y = y_0 + a(x-1)^p + \cdots$ $(p > 0)$ to (A1d), we can also show that for $\delta < 0$ or $\delta > 2$ there exists no other solution that reaches the point $(1, y_0)$ for $y_0^2 < 1$, while for $0 < \delta < 1$ or $1 < \delta \leq 2$ there exist one parameter family of such solutions with $p = (1 + 2\delta - \delta^2)/2$. Hence, for the latter case, one can reach every point on the open segment I from a point with $x > 1$ by a null geodesic. Fig. 3 gives two examples of such a family of null geodesics for the case $\delta = 2$.

The curvature invariants also exhibit singular behavior at the two points $\mathrm{P}_\pm$: $(\rho, z) = (0, \pm\sigma)$. To see this, we introduce the new coordinate system (X, Y) defined by

$$X^2 = \frac{1-y^2}{x^2-1}, \quad Y = \frac{y}{x}. \tag{11}$$

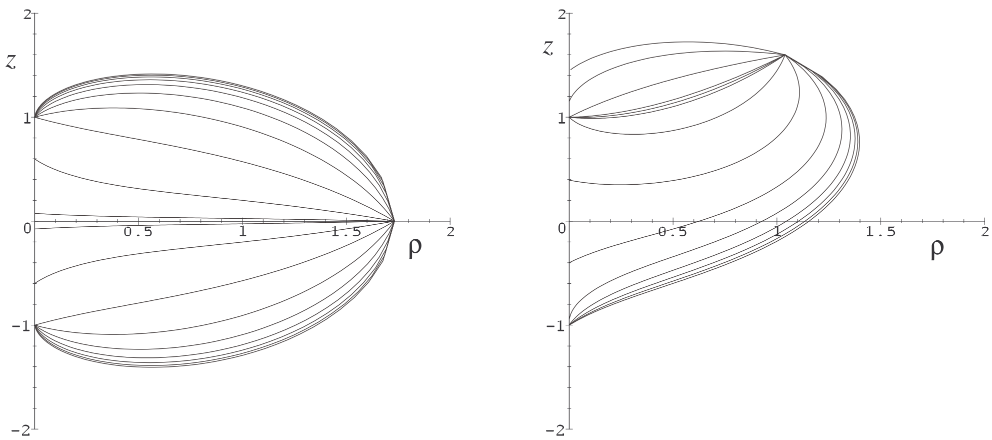

FIG. 3: Two examples of a family of null geodesics reaching the singular segment I from the same point for $\delta = 2$. These figures are drawn in the units $\sigma = 1$.

The coordinates x and y are expressed in terms of X and Y as

$$x = \sqrt{\frac{X^2+1}{X^2+Y^2}}, \quad y = Y\sqrt{\frac{X^2+1}{X^2+Y^2}}. \tag{12}$$

The region $x \geq 1, |y| \leq 1$ is mapped to the region $X \geq 0, |Y| \leq 1$, and X =const curves and Y =const curves are represented by circles with centers on the ρ-axis and on the z-axis, respectively, as shown in Fig.2.

Inserting these expressions into (9), we find that the curvature invariants behave in the limit $Y^2 \to 1$ with fixed X as

$$\Psi_2 \approx \frac{\delta(1-\delta)}{2^{2\delta+1}}\left(1+X^2\right)^{\delta^2-\delta+1}(1-Y^2)^{\delta-2}, \tag{13a}$$

$$\Psi_0\Psi_4 \approx \frac{\delta^2(\delta-1)^2}{2^{4\delta+4}}\left(1+X^2\right)^{2\delta^2-2\delta+1}\left(9+(2\delta-1)^2X^2\right)\left(1-Y^2\right)^{2(\delta-2)}. \tag{13b}$$

From this, we see that the two points $\mathrm{P}_\pm$(i.e., $Y = \pm 1$) are curvature singularities for $\delta < 2$ ($\delta \neq 0, 1$). In this case, one can reach each of these points by a spatial curve as well as by a null geodesic along the z-axis in the $\rho - z$ plane with a finite affine length ℓ, because it follows from (A1c) that the affine parameter is proportional to the x-coordinate for such geodesics. Hence, they are naked.

In contrast, for $\delta > 2$, the curvatures vanish as one approaches $\mathrm{P}_\pm$. Hence, they are not curvature singularity. Furthermore, there exists no spatial curve that reaches these points within a finite proper length, as was pointed out by Papadopoulos et al[6]. However, since null geodesics along the z-axis reach these points in finite affine parameters, they are quasi-regular singularity for the region $x > 1$[6]. In the next subsection, we will show that these points are in reality Killing horizons at least for $\delta = 3$ and $\delta \geq 4$.

The situation for $\delta = 2$ is similar but more subtle. The curvature invariants approach non-vanishing values in the limit $Y \to \pm 1$, and these limits depend on X, i.e., on the direction one approaches the points $\mathrm{P}_\pm$. They are again at spatial infinity, but can be reached by null geodesics with finite affine parameters. Hence, they are apparently directional quasi-regular singularities[6]. We will also show that these points are Killing horizons.

Next, let us examine geometrical shapes of these singularities. One simple quantity characterizing shapes is the circumferential radius R of circles generated by the Killing vector $\eta = \partial_\phi$, given

δ	Curvature		Radius		Length			Proper distance		Affine distance
	I	P$_\pm$	I	P$_\pm$	L_x	L_X	L_n	I	P$_\pm$	I/P$_\pm$
$\delta < 0$	∞	∞	0	0	0	0	0	finite	finite	finite
$\delta = 0$	0	0	0	0	2σ	2σ	2σ	finte	finite	finite
$0 < \delta < 1$	∞	∞	0	0	∞	∞	∞	finite	finite	finite
$\delta = 1$	finite	finite	0	finite	$2\pi\sigma$	$2\pi\sigma$	$2\pi\sigma$	finite	finite	finite
$1 < \delta < 2$	∞	∞	∞	0	0	0	0	finite	finite	finite
$\delta = 2$	∞	finite	∞	$4\sigma X$	8σ	∞	0	finite	∞	finite
$\delta > 2$	∞	0	∞	∞	∞	∞	0	finite	∞	finite

TABLE I: Asymptotic behavior of the ZVW spacetime near the singularity. The second and third columns show the limits of the curvature invariants on the open segment I: $\rho = 0, -\sigma < z < \sigma$ and at the two points P$_\pm$: $(\rho, z) = (0, \pm\sigma)$. The next two columns with the heading 'Radius' show the limit of R on the open segment I and at the two points P$_\pm$, respectively, and the following three columns with the heading 'Length' show the limit at $x = 1$ of the length of the curves in the $x - y$ plane with $x =$const, $X =$const and $X(1 - Y^2)^n =$const, respectively. Finally, the two columns denoted by 'Proper distance' show the proper length ℓ of spatial curves that approach a point on I and the two points P$_\pm$, respectively, and the last column shows the behaviour of the affine length of null geodesics approaching the singularities.

by (5). The asymptotic behavior of this quantity near the singularity is summarized in Table I. This asymptotic behavior suggests that the singularity is point-like or string like for $\delta < 1$ ($\delta \neq 0$) and is ring-like for $\delta > 1$.

To confirm this, we estimate the length of the singularity in the z-direction. There are various candidates for the definition of such a length. One natural candidate is the length of $x =$const curves,

$$\begin{aligned} L_x &:= \int_{-1}^{1} \frac{\Sigma dy}{\sqrt{1 - y^2}} \\ &= \sigma(x+1)^{\delta(\delta+1)/2}(x-1)^{\delta(\delta-1)/2} \int_0^{\pi/2} d\theta (x^2 - 1 + \sin^2\theta)^{(1-\delta^2)/2}. \end{aligned} \tag{14}$$

The second candidate is the length of $X =$const curves, which is expressed in the limit $X \to +\infty$ as

$$L_X \approx 2^\delta \sigma X^{\delta(1-\delta)} \int_{-1}^{1} \frac{dY}{(1 - Y^2)^{\delta/2}}. \tag{15}$$

Here, the integral on the right-hand side diverges for $\delta \geq 2$, hence L_X can have a finite limit only for $\delta < 2$. The asymptotic behavior of these quantities in the limits $x \to 1$ and $X \to +\infty$ are summarized in Table I. Comparison of this behavior of L_x and L_X with that of R shows that the singularity is point-like for $\delta < 0$, string-like for $0 < \delta < 1$, and ring-like for $1 < \delta < 2$. In contrast, these quantities do not seem to describe the length of singularities for $\delta \geq 2$, because the two points P$_\pm$ are horizons and not singularity, as we will show later. For these cases, we have to measure the length of the singularity by the limit length of curves which are tangent to the segment I at $(\rho, z) = (0, \pm\sigma)$. Therefore, we define the length of I by the $a \to \infty$ limit of the length L_n of curves with $X(1 - Y^2)^n = a$(constant) for $n > 0$. In this limit, L_n is expressed as

$$L_n \approx 2^\delta \sigma a^{\delta(1-\delta)} \int_{-1}^{1} dY\, (1 - Y^2)^{n\delta(\delta-1)-\delta/2}. \tag{16}$$

The integral on the right-hand side of this equation converges for $2 - \delta - 2n\delta(1 - \delta) > 0$. For any δ, there exists $n > 0$ that satisfies this inequality, and for such n, the asymptotic behavior of L_n is

simply determined by the sign of $\delta(1-\delta)$, as summarized in Table I. This behavior implies that the singularity is ring-like for $\delta \geq 2$.

To summarize, the segment singularity I is geometrically point-like for $\delta < 0$, string-like for $0 < \delta < 1$ and ring-like for $\delta > 1$. This feature is consistent with the behavior (7) of the metric at spatial infinity for $\delta > 0$, although such a good correspondence does not exist for $\delta < 0$.

Here, note that this characterization of the singularity is based on the purely local geometry and may not coincide with its physical appearance. For example, if an observer in the region $x > 1$ looks at the singularity by light rays, he will observe a cylinder-like object for the case $1 < \delta \leq 2$, because light rays starting from different points on I can reach the same point in $x > 1$. For the other values of δ, the geometrical shape and the physical appearance coincide.

Finally, we examine the Komar mass of the singularity. The Komar mass is in general defined in terms of the time-like Killing vector $\xi = \partial_t$ as[18–20]

$$M = -\frac{1}{16\pi}\int_\Sigma \epsilon_{\mu\nu\lambda\sigma}\nabla^\mu \xi^\nu dx^\lambda \wedge dx^\sigma. \tag{17}$$

Here, the orientation of the spatial two-surface Σ should be chosen so that for a future-directed time-like vector uCan outward normal n to Σ and a basis (v,w) giving the orientation of Σ, (u,n,v,w) gives the positive orientation of the spacetime.

For the ZVW metric, in the case in which Σ is an axisymmetric surface determined by a curve γ in the $x-y$ plane, this expression reduces to

$$M = \frac{\delta\sigma}{2}\int_\gamma dy, \tag{18}$$

where the orientation of γ should be taken so that for the tangent vector V of γ and a normal N to γ directed outward from the region enclosed by γ, (N,V) has the same orientation as that of (∂_x, ∂_y). It is clear from this expression that each portion of the segment singularity has a positive mass and the total mass carried by the singularity coincides with the total mass of the system, $M = \delta\sigma$. This result is quite interesting because most of the known examples of naked singularities have negative or zero mass[13, 21]. For example, central shell-focusing singularities formed by spherically symmetric gravitational collapse always have non-positive local gravitational mass if they are locally naked[17]. One exception is a shell-crossing singularity, which can have a positive mass. However, this type of singularity is weak in the sense that the spacetime has a C^0 extension across the singularity[22]. Another counter example is the time-like singularity of the Janis-Newman-Winicour solution[23, 24] for the spherically symmetric Einstein-Massless-Scalar (EMS) system. Virbhadra showed that this singularity has a positive mass if one defines the mass by the spatial integration of the Einstein energy-momentum complex[25]. This mass is identical to the Komar mass, whose value is independent of the location of the integration surface Σ for a static solution of the spherically symmetric EMS system. However, it does not coincide with the local gravitational mass used in the mass inflation argument[26]. We can show that the latter becomes negatively infinite for the singularity of the Janis-Newman-Winicour solution. This discrepancy seems to come from the difference in the matter contribution to the mass. In contrast, because the ZVW solution discussed in the present paper is a vacuum solution, this kind of ambiguity associated with the matter contribution does not exist in the definition of the mass for the singularity of the ZVW spacetime. Thus, it seems to provide a clearer non-trivial example of naked singularities with positive gravitational mass.

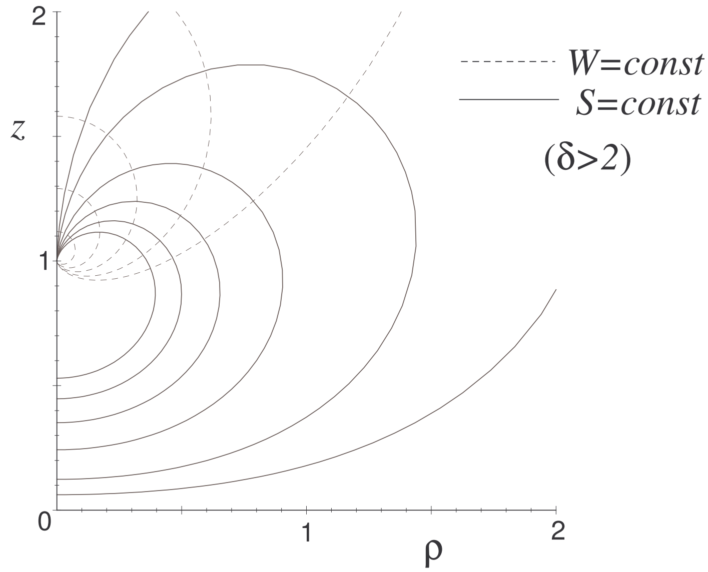

FIG. 4: The relation between the (S, W)-coordinates and (ρ, z)-coordinates for $\delta > 2$. The coordinate values of ρ and z are given in the units $\sigma = 1$.

B. Extension across $\mathrm{P}_\pm$

In this subsection, we will demonstrate that the two points $\mathrm{P}_\pm$ are Killing horizons for $\delta = 2, 3$ and $\delta \geq 4$, as anticipated by Papadopoulos et al[6], by constructing smooth extensions of the spacetime across these points.

The coordinate system around the points $\mathrm{P}_\pm$ $((X, Y) = (0, \pm 1))$ that we adopt to extend the spacetime is

$$S = 2^\delta \sigma \frac{X}{\sqrt{1+X^2}} \left(\frac{1-Y^2}{Y^2}\right)^{(2-\delta)/2}, \tag{19a}$$

$$W = \frac{1}{4}(1-Y^2)(1+X^2)^{\delta-2}. \tag{19b}$$

For $\delta = 2$, S and W depend only on X and Y, respectively, and these two coordinate systems are essentially equivalent. However, for $\delta > 2$, each $S =$const curve is tangential to the z-axis in the $\rho - z$ plane, as is shown in Fig. 4. Hence, the (S, W) coordinates expand cusp regions at $\mathrm{P}_\pm$ in the (X, Y) coordinates. Such a singular coordinate system is required because R diverges as Y tends ± 1 along $X =$const curves for $\delta > 2$, as is shown in Table I.

In this new coordinate system, the ZVW metric is expressed as

$$ds^2 = -aW^\delta \left(dt^2 - h^2 \frac{dW^2}{W^{2\delta}}\right) + bS^2 d\phi^2 + cdS^2, \tag{20}$$

with

$$a = \left(\frac{2}{x+1}\right)^{2\delta} (X^2+Y^2)^{-\delta}(1+X^2)^{\delta(2-\delta)}, \tag{21a}$$

$$b = \left(\frac{x+1}{2}\right)^{2\delta} Y^{2(2-\delta)}(1+X^2)(X^2+Y^2)^{\delta-2}, \tag{21b}$$

$$c = \left(\frac{x+1}{2}\right)^{2\delta} \frac{(1+X^2)^{3-\delta^2}(X^2+Y^2)^{\delta-2}}{Y^{2(\delta-3)}[Y^2+(\delta-2)^2X^2]}, \tag{21c}$$

$$h^2 = 4\sigma^2\left(\frac{x+1}{2}\right)^{4\delta} \frac{(1+X^2)^{\delta^2-6\delta+6}(X^2+Y^2)^{2(\delta-1)}}{Y^2+(\delta-2)^2X^2}. \tag{21d}$$

Here, the dependence of x, X and Y on S and W is implicitly determined by the relations

$$x = \sqrt{\frac{X^2+1}{X^2+Y^2}}, \tag{22a}$$

$$Y^2 = 1 - \frac{4W}{(1+X^2)^{\delta-2}}, \tag{22b}$$

$$X^2 = \frac{S^2}{16\sigma^2}\frac{(1+X^2)W^{\delta-2}}{[(1+X^2)^{\delta-2}-4W]^{\delta-2}}. \tag{22c}$$

This coordinate system covers only a neighbourhood of P_+ or P_- in general, and $P_\pm$ corresponds to $W=0$. For $\delta=2$, a, b and c have non-vanishing finite limits at $W=0$, which are expressed as regular functions of X, or equivalently of S. On the other hand, for $\delta>2$, in the limit $W\to 0$ with S fixed, X^2 behaves as

$$X^2 = \frac{S^2}{16\sigma^2}W^{\delta-2}\left(1+\mathrm{O}(W)\right). \tag{23}$$

Hence, a, b and c approach unity in the same limit.

Here, note that if δ is an integer equal to or greater than 2, a, b and c have unique regular analytic extensions to the region $W<0$. A C^2 extension is possible even for non-integer δ by replacing W^δ in (22c) by $|W|^\delta$, if $\delta\geq 4$.

In the extension of the spacetime, a crucial point is the asymptotic behavior of h at $W=0$. First, we examine it for $\delta>2$. Let us define the function $h_0(W)$ by

$$h_0(W) := h(S=0,W) = \frac{2\sigma}{(1-4W)^{3/2}}\left[\tfrac{1}{2} - 2W + W^2 + \tfrac{1}{2}(1-2W)\sqrt{1-4W}\right]^{\delta/2}. \tag{24}$$

Then, we can show that $h^2-h_0^2$ is expressed as

$$\begin{aligned} h(S,W)^2 - h_0(W)^2 &= \left[16\delta(\delta-1)W^2+\mathrm{O}\big(W^3\big)\right]\sigma^2X^2 \\ &\quad -2(\delta-1)(\delta-2)^2(\delta-3)\sigma^2X^4 + \mathrm{O}\big(WX^4, X^6\big) \\ &= \delta(\delta-1)W^\delta S^2 - \frac{(\delta-1)(\delta-2)^2(\delta-3)}{128\sigma^2}W^{2\delta-4}S^4 \\ &\quad +\mathrm{O}\Big(W^{\delta+1}, W^{2\delta-3}\Big). \end{aligned} \tag{25}$$

From this, it follows that for $\delta=3$ or $\delta\geq 4$, h^2 is expressed in terms of a function $k(S,W)$ that is continuous at $W=0$ as

$$h^2(S,W) = h_0^2(W) + W^\delta k(S,W). \tag{26}$$

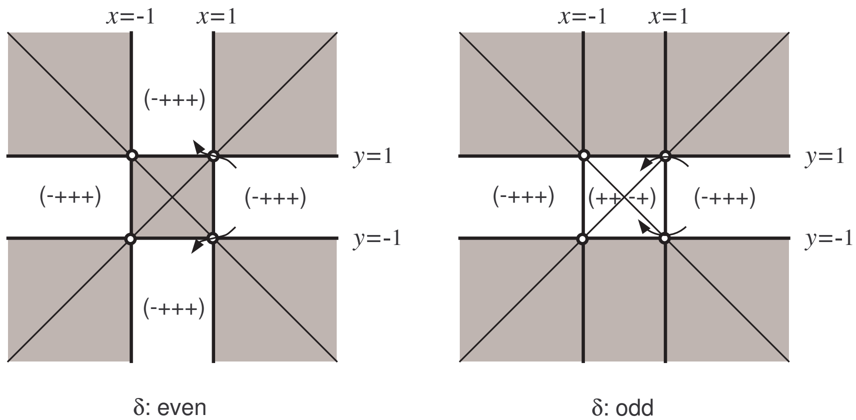

FIG. 5: The signature of the metric in the $x-y$ plane for the case δ is an integer, and the path of the spacetime extension across $\mathrm{P}_\pm$: $(x,y)=(1,\pm 1)$. In the shaded regions, the metric has the wrong signature $(+---)$ or its permutation.

Next, for $\delta=2$, h^2 can be expanded with respect to W as

$$h^2 = 4\sigma^2\left[1+4W+\left(20-\frac{S^2}{2\sigma^2}+\frac{S^4}{64\sigma^4}\right)W^2+\mathrm{O}\big(W^3\big)\right]. \tag{27}$$

Hence, the expression (26) also holds in this case with a function k that is smooth at $W=0$.

Now, we can construct a regular extension across $\mathrm{P}_\pm$ by introducing the retarded/advanced-type coordinate

$$u_\pm := t \pm \int^W h_0(W)\frac{dW}{W^\delta}. \tag{28}$$

In term of the coordinate system $(t,\phi,u_\pm,S)$, the ZVW metric is written

$$ds^2 = -aW^\delta du_\pm^2 \pm 2ah_0 dW du_\pm + akdW^2 + bS^2d\phi^2 + cdS^2. \tag{29}$$

It is clear from the comment above on the behavior of a, b, c and k that this metric can be regularly extended to the region $W<0$ for $\delta=2,3$ and $\delta\geq 4$. Here, it is understood that W^δ is replaced by $|W|^\delta$ for non-integer δ, as mentioned before. This extension connects the regular solutions in two regions with $x>1$ and $-1<x<1$ as shown in Fig. 5, and is analytic and unique in the case δ is an integer (the extension path in the $x-y$ plane for a non-integer δ is the same as that in the case δ is an even integer). In contrast, for non-integer δ such that $\delta>4$, the extension is at least $C^{[\delta]-2}$, where $[\delta]$ represents the maximum integer less than or equal to δ, but the $([\delta]-1)$-th derivatives of the metric are divergent at $W=0$ in general.

Thus, we have shown that the ZVW spacetime with $\delta=2,3$ or $\delta\geq 4$ has Killing horizons at $\mathrm{P}_\pm$: $(\rho,z)=(0,\pm\sigma)$. These two horizons share the ring singularity corresponding to the open segment I as the common boundary. As a whole, they can be regarded as a single spherical horizon with a ring singularity at the equator whose circumference is infinite, as is illustrated in Fig. 6. Since the W-derivative of g_{tt} vanishes at $W=0$, this horizon is degenerate in its regular part in the sense that the surface gravity vanishes.

Here, note that the four-dimensional extension constructed here coincides with the two-dimensional extension along the z-axis constructed by Papadopoulos et al[6]. In particular, the conformal diagram of this two-dimensional sector of the extended spacetime is given by Fig. 3 for

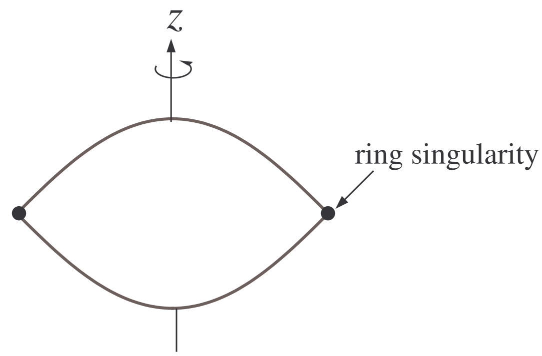

FIG. 6: The shape of a spatial section of the horizon for the $\delta = 2$ ZVW spacetime, conformally embedded in the Minkowski spacetime $E^{2,1}$. The ring singularity marked by a full circle actually has an infinite circumference.

even δ and by Fig. 4 for odd δ in their paper. We can also easily draw a conformal diagram of the two-dimensional sector corresponding to the equatorial plane. The result is just the same as the diagram for a negative-mass Schwarzschild black hole, for any $\delta \neq 1$. Hence, the singularity at the segment I is time-like.

Finally, we comment on the area of the horizon. From (29), its area A for $\delta > 2$ is infinite:

$$A = \lim_{W \to 0} 2 \times 2\pi \int_0^\infty \sqrt{b} S dS = +\infty. \tag{30}$$

In contrast, for $\delta = 2$, it is given by the finite value

$$A = \lim_{W \to 0} 2 \times 2\pi \int_0^{4\sigma} \sqrt{bc} S dS = 32\pi\sigma^2. \tag{31}$$

This horizon area for the $\delta = 2$ case is half of that for the Schwarzschild black hole with the same mass $M = 2\sigma$ and equal to twice of that for the Schwarzschild black hole with mass $M/2$. To be precise, we can show that the Schwarzschild solution with mass $M = 2\sigma$ can be represented by the Israel-Kahn solution[27] with two black holes of the same mass whose centers are separated by 2σ in the Weyl coordinates, and when these centers come closer, the total horizon area decreases monotonically. The above area of the $\delta = 2$ ZVW solution is obtained in the limit the two centers coincide.

III. TOMIMATSU-SATO SPACETIME

In this section, we extend the analysis on the ZVW solution to the Tomimatsu-Sato solution. In the present paper, we only consider TS2, i.e., the $\delta = 2$ case.

The metric of TS2 is expressed in the prolate spheroidal coordinates as[1]

$$ds^2 = -f dt^2 + 2f\omega dt d\phi + R^2 d\phi^2 + \Sigma^2 \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right), \tag{32}$$

with

$$f = \frac{A}{B}, \quad f\omega = \frac{4\sigma q}{p} \frac{(1 - y^2)C}{B}, \quad \Sigma^2 = \frac{B}{p^4 (x^2 - y^2)^3}, \quad R^2 = \frac{\sigma^2}{p^2} \frac{(1 - y^2)D}{B}. \tag{33}$$

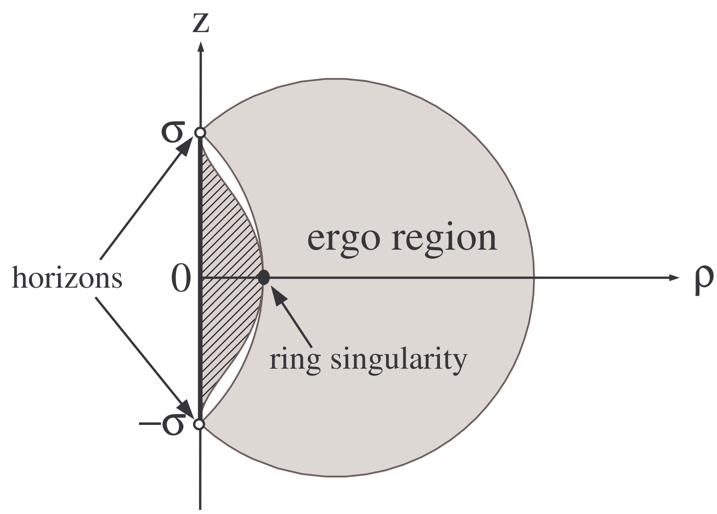

FIG. 7: The structure of the Tomimatsu-Sato spacetime with $\delta = 2$ and $p = 1/\sqrt{2}$. In the hatched gray region, the rotational Killing vector ∂_ϕ becomes time-like and its orbits give closed timelike curves.

Here, A, B, C and D are polynomials given by

$$A = (1-y^2)^4 g(Z) g(-Z), \tag{34a}$$

$$B = (g(x) + q^2y^4)^2 + 4q^2y^2\left[px^3 + 1 - (px+1)y^2\right]^2, \tag{34b}$$

$$\begin{aligned} C = \ & q^2(px+1)y^4(-y^2+3) + \left\{-2q^2(px^3+1) + (px+1)g(x)\right\}y^2 \\ & -(2px^3 - px + 1)g(x), \end{aligned} \tag{34c}$$

$$\begin{aligned} D = \ & p^2q^2(x^2-1)y^8 + 4q^2(px+1)(p^3x^3 + 3p^2x^2 - p^3x + 4px - 3p^2 + 4)y^6 \\ & -2q^2(3p^4x^6 + 4p^3x^5 - 3p^4x^4 + 8p^3x^3 + 37p^2x^2 - 12p^3x + 48px - 13p^2 + 24)y^4 \\ & +4q^2(p^4x^8 - p^4x^6 + 4p^3x^5 - 4p^3x^3 + 15p^2x^2 + 24px - 3p^2 + 12)y^2 \\ & +g(x)(p^4x^6 + 6p^3x^5 - p^4x^4 + 16p^2x^4 - 12p^3x^3 + 32px^3 \\ & \quad +15p^2x^2 + 6p^3x - 15p^2 + 16), \end{aligned} \tag{34d}$$

where

$$g(x) = p^2x^4 + 2px^3 - 2px - 1, \quad Z^2 := \frac{x^2 - y^2}{1 - y^2}. \tag{35}$$

This solution is asymptotically flat, and the parameters σ, p and q are related to its mass M and angular momentum J as

$$p^2 + q^2 = 1, \quad M = \frac{2}{p}\sigma, \quad J = M^2 q. \tag{36}$$

In particular, in the non-rotating case $q = 0$, this solution reduces to the $\delta = 2$ ZVW solution, which is referred to as ZVW2 from this point. In the present paper, we only consider the case $0 < p, q < 1$.

A. Basic properties

Although TS2 reduces to ZVW2 in the limit $q = 0$, the former solution has quite different features compared with the latter[2–4].

1. Ergo region

$f = -g_{tt} = -\xi \cdot \xi$ is not positive definite in the asymptotically flat region $x > 1, -1 \leq y \leq 1$, and the so-called ergo region appears[4]. From (34a) and (34b), f becomes negative in the region $A < 0$. Since $g(x) = 0$ has two single real roots $-x_1$ and x_0 with $1 < x_0 < x_1$ for $0 < p < 1$, the ergo region in $x \geq 1$ is given by

$$x_0^2 - (x_0^2 - 1)y^2 < x^2 < x_1^2 - (x_1^2 - 1)y^2. \tag{37}$$

Note that since Z in (35) is expressed in terms of the (X, Y) coordinates in (11) as $Z^2 = 1 + 1/X^2$, the boundaries of this ergo region, i.e., the infinite redshift surfaces, are represented as $X =$const (see Fig. 7).

2. Causality violation

Since D is not positive definite in $x \geq 1$, the norm R^2 of the rotation Killing vector $\eta = \partial_\phi$ becomes negative in some region[4]. Each S^1 orbit in this region becomes a closed time-like curve. This region contains the segment I: $\rho = 0, -\sigma < z < \sigma$, where ρ and z are the Weyl-type coordinates related to x and y by (3), and contacts with the inner boundary of the ergo region from (34d), as shown in Fig. 7. The appearance of this causality violating region in the domain of outer communication is the most exotic feature of TS2.

3. Singularity

From (32), TS2 may have curvature singularity at points where $B = 0$, in addition to the points $x = \pm 1$ and $x = \pm y$ in the Weyl case. This is confirmed by looking at the behavior of the Weyl curvature. As in the case of the ZVW solutions, $\Psi_1 = \Psi_3 = 0$ with respect to the null complex tetrad (8) with dt replaced by $dt - \omega d\phi$, and all polynomial curvature invariants are expressed as polynomials of Ψ_2 and $\Psi_0\Psi_4$. By symbolic computations, we can show that they have the forms

$$\Psi_2 = \frac{P_1}{B^3}, \quad \Psi_0\Psi_4 = \frac{P_2}{B^6}, \tag{38}$$

where P_1 and P_2 are polynomials of x and y that have no non-trivial common divisor with B. Since the explicit expressions for these curvature invariants are quite long, we only give their values at the equatorial plane $y = 0$:

$$\Psi_2(x, y = 0) = \frac{p^4 x^6(-2 - 3px + px^3)}{g(x)^3}, \tag{39a}$$

$$(\Psi_0\Psi_4)(x, y = 0) = \frac{9p^8 x^{10}(x^8 p^2 - 2p^2 x^6 + 4x^5 p + 5p^2 x^4 - 4px^3 + 8px + 4)}{g(x)^6}. \tag{39b}$$

From these expressions, it follows that the curvature invariants diverge at the zero of $g(x)$, $x = x_0$, which is just the intersection of the inner boundary of the ergo region with the equatorial plane. It is easy to see from (34b) that B vanishes only at $(x, y) = (x_0, 0)$ for $x \geq 1$ and $0 < p < 1$, except for the two points $\mathrm{P}_\pm$: $(x, y) = (1, \pm 1)$. Hence, in contrast to the ZVW spacetime, there exists no curvature singularity on the open segment I, and instead there appears the famous ring singularity[1], which has an infinite circumference, because $x = x_0$ is a single root of D and a double root of B on the equatorial plane $y = 0$ from (34b) and (34d). This ring singularity reduces to the ring-like singularity at the segment I of ZVW2 in the limit $q \to 0$.

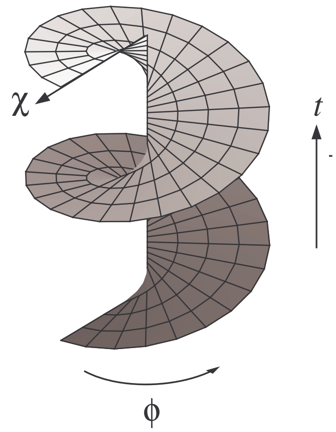

FIG. 8: $\tilde{t}$ =const surfaces

B also vanishes at $\mathrm{P}_\pm$. However, we can show by symbolic computations that the curvature invariants have finite limits at these points. These limits are direction dependent as in the case of the ZVW spacetime. For example, their values in the cases in which one approaches $\mathrm{P}_\pm$ from $|z| > \sigma$ and from $|z| < \sigma$ along the z-axis are given by

$$X = 0, Y = \pm 1 : \quad \Psi_2 = -\frac{p(p \pm iq)}{8(1+p)}, \quad \Psi_0 \Psi_4 = \frac{9p^2(p \mp iq)^2}{64(p+1)^2}, \tag{40a}$$

$$X = \infty, Y = \pm 1 : \quad \Psi_2 = \frac{p^4(-q \pm ip)}{8q^3(1+p)}, \quad \Psi_0 \Psi_4 = -\frac{9p^8(p \mp iq)^2}{64q^6(p+1)^2}. \tag{40b}$$

Thus, $\mathrm{P}_\pm$ are quasi-regular singularities[4, 5]. In the next subsection, we will show that these two points are disconnected Killing horizons, by constructing regular extensions of the spacetime across these points.

All metric components of (32) as well as the polynomial curvature invariants are regular and finite on the open segment I, which is expressed as $x = 1, -1 < y < 1$ in the (x, y) coordinates, in contrast to the case of ZVW2. However, the spacetime turns out to be singular on this segment[4]. To see this, let us rewrite the metric (32) as

$$ds^2 = -f(dt - \omega d\phi)^2 + \Sigma^2 \left(d\chi^2 + \ell^2 \sinh^2 \chi d\phi^2 \right) + \Sigma^2 \frac{dy^2}{1-y^2}, \tag{41}$$

where χ is related to x by $x = \cosh\chi$, and ℓ is defined by

$$\ell^2 = \frac{\sigma^2(1-y^2)}{f\Sigma^2}. \tag{42}$$

In the neighbourhood of the segment I, ω behaves as

$$\begin{aligned} \omega &= \frac{4\sigma(p+1)}{qp} + \omega_1(x, y)\sinh^2\chi; \\ \omega_1 &= -\frac{4\sigma(p+1)[(p-1)(y^4+1) + 2(3p+1)y^2]}{q^3(1-y^2)^2} + \mathrm{O}(\chi^2)\,. \end{aligned} \tag{43}$$

Hence, if we introduce the new time coordinate $\tilde{t}$ by

$$\tilde{t} = t - \omega_0 \phi; \quad \omega_0 = \frac{4\sigma(p+1)}{qp}, \tag{44}$$

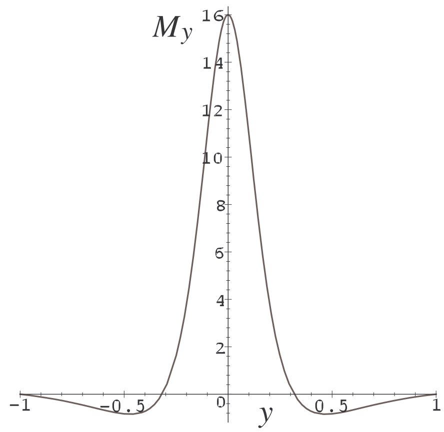

FIG. 9: The behavior of M_y at $x = 1$ for $p = 0.5$

(41) can be written as

$$ds^2 = -f(d\tilde{t} - \omega_1 \sinh^2 \chi d\phi)^2 + \Sigma^2 \left(d\chi^2 + \ell^2 \sinh^2 \chi d\phi^2\right) + \Sigma^2 \frac{dy^2}{1-y^2}. \tag{45}$$

From this, it follows that the spacetime is regular on the segment I, if ℓ is unity on I. However, since ℓ behaves around I as

$$\ell = \frac{p^2}{q^2} + \frac{p^2(p^2+3)}{q^4(1-y^2)} \sinh^2 \chi + \mathrm{O}\left(\chi^4\right), \tag{46}$$

the spacetime is not regular on I unless $p = q = 1/\sqrt{2}$ and has a conical singularity there. The deficit angle is positive for $q^2 > p^2$ and negative for $q^2 < p^2$.

This is not the whole story. There is another subtlety concerning the spacetime structure around I[4]. Since ϕ is an angle coordinate and defined modulo 2π, the new time coordinate $\tilde{t}$ is not a single-valued function, and each $\tilde{t}$ =const surface has a helical structure in the original coordinates, as illustrated in Fig. 8. These helical surfaces cannot be extended regularly to the axis $\chi = 0$. Hence, ϕ cannot be regarded as an angle coordinate on these $\tilde{t}$ =const surfaces, but rather should be regarded as a parameter of helical orbits of the Killing vector $\partial_\phi + \omega_0 \partial_t$. In this sense, the deficit angle interpretation above does not have a global meaning. This exotic structure arises because the two Killing vectors ∂_t and ∂_ϕ become parallel on I.

4. Komar mass

For the metric (32), the Komar mass (17) contained in a 3-surface that is bounded by a closed 2-surface determined by a curve γ in the $x - y$ plane is expressed as

$$M = \int_\gamma (M_x dx + M_y dy), \tag{47}$$

where

$$(1-y^2) M_y dx - (x^2-1) M_x dy = \frac{1}{4\sigma}\left[R^2 df + f\omega d(f\omega)\right]. \tag{48}$$

We can show that for any curve γ whose end points are on the regular part of the z-axis, $z > \sigma$ and $z < -\sigma$, M is equal to the total mass $M = 2\sigma/p$, if γ does not pass the ring singularity. This

implies that the ring singularity has zero gravitational mass. In contrast, from the asymptotic behavior of M_x and M_y around $x = 1$,

$$M_x = \frac{4\sigma y[(1-p)(y^6-y^4)+(p+7)y^2+9-p]}{[(1-p)y^4+2(p+3)y^2+1-p]^2} + \mathrm{O}(x-1)\,, \tag{49a}$$

$$M_y = \frac{4\sigma(1-y^2)[(1-p)(y^4+1)-2(3p+1)y^2]}{p[(1-p)y^4+2(p+3)y^2+1-p]^2} + \mathrm{O}(x-1)\,, \tag{49b}$$

the main contribution to the integral on the right-hand side of (47) comes from the segment I for a curve close to I (see Fig. 9). This seems to indicate that the gravitational mass is carried by the singularity on I. However, this interpretation may not be valid, because a 2-surface corresponding to a curve close to the segment I is time-like. Furthermore, M_y is not positive definite, as shown in Fig. 9.

B. Extension across $\mathrm{P}_\pm$

In order to construct an extension of TS2, we utilize the coordinates (X, W), where X is the coordinate defined in (11) and W is defined in terms of the Y coordinate there by

$$W = \frac{1}{4}(1-Y^2). \tag{50}$$

Then, we can show that the metric is written in terms of functions $a(X, W)$, $b(X, W)$, $c(X, W)$, $h(X, W)$ and $\Omega(X, W)$ that are regular analytic in a neighbourhood of $W = 0$i$Y = \pm 1$) as

$$ds^2 = -aW^2\left(dt^2 - h^2\frac{dW^2}{W^4}\right) + bX^2(d\phi - \Omega dt)^2 + cdX^2. \tag{51}$$

The asymptotic behavior of these functions at $W = 0$ are given by

$$a = \frac{2(p^2+q^2X^4)}{(p+1)(1+X^2)^2} + \frac{8(p^2-q^2X^6)}{(p+1)(1+X^2)^3}W + \mathrm{O}\big(W^2\big)\,, \tag{52a}$$

$$b = \frac{8\sigma^2(p+1)}{p^2+q^2X^4} + \mathrm{O}(W)\,, \tag{52b}$$

$$c = \frac{8\sigma^2(p+1)(p^2+q^2X^4)}{p^4(1+X^2)^4} + \mathrm{O}(W)\,, \tag{52c}$$

$$\begin{aligned} h^2 = {} & \frac{(p+1)^2}{p^4}\left\{1+4W+4(p+4)W^2\right\} \\ & - \frac{4(p+1)^2\left\{2p^3+p^2(2p-1)X^2+q^2X^4\right\}X^2}{p^4(1+X^2)^2}W^2 + \mathrm{O}\big(W^3\big)\,, \end{aligned} \tag{52d}$$

$$\sigma\Omega = \frac{pq}{p+1}W - \frac{q(p^2+q^2X^4)X^2}{p(p+1)(1+X^2)^2}W^2 + \mathrm{O}\big(W^3\big)\,. \tag{52e}$$

From this behavior, if we put

$$h_0(W) = h(X=0, W), \quad \Omega_0(W) = \Omega(X=0, W), \tag{53}$$

h^2 and Ω can be expressed in terms of functions $k(X, W)$ and $\Omega_1(X, W)$ that are regular analytic at $W = 0$ as

$$h^2 = h_0(W)^2 + k(X, W)W^2, \tag{54a}$$

$$\Omega = \Omega_0(W) + \Omega_1(X, W)W^2. \tag{54b}$$

Hence, in the advanced/retarded coordinates

$$u_\pm := t \pm \int^W h_0(W)\frac{dW}{W^2}, \tag{55a}$$

$$\phi_\pm := \phi \pm \int^W \Omega_0(W)h_0(W)\frac{dW}{W^2}, \tag{55b}$$

the metric takes the form that is regular analytic at $W = 0$:

$$ds^2 = -aW^2du_\pm^2 \pm 2h_0du_\pm dW + akdW^2 + bX^2(d\phi_\pm \pm \Omega_1 h_0 dW - \Omega du_\pm)^2 + cdX^2. \tag{56}$$

Since $W = 0$ is a null hypersurface and the Killing vector $\xi = \partial_t$ is its null tangent, the hypersurface $W = 0$ is a Killing horizon. Because $W = 0$ is the double root of ρ^2/R^2, the horizon is degenerate, i.e., the surface gravity vanishes. Further, since Ω vanishes at $W = 0$, the horizons are non-rotating.

C. Area and shape of the horizons

It is easy to estimate the area of the horizons. From (52) and (56), we obtain

$$A = 2 \times \lim_{W\to 0} 2\pi \int_0^\infty \sqrt{bc}XdX = \frac{16\pi\sigma^2(p+1)}{p^2}, \tag{57}$$

which coincides with (31) for the non-rotating limit $p = 1$. Since it can be written as

$$A = 4\pi M(M + \sqrt{M^2 - a^2}) \tag{58}$$

with $a = J/M$, it is half of the area of the Kerr black hole with the same mass M.

Although the horizon area was finite in spite of its non-compactness in the case of ZVW2, the horizons become compact in the case of TS2. In order to see this, let us rewrite the metric of the two-dimensional section $W = 0, u_\pm =$const of the horizon as

$$ds_H^2 = \frac{2\sigma^2(p+1)}{p^2\ell(\theta)}\left[d\theta^2 + \ell^2(\theta)\sin^2\theta d\phi_\pm^2\right], \tag{59}$$

where

$$X = \tan\frac{\theta}{2}\ (0 \le \theta \le \pi), \tag{60}$$

$$\ell(\theta) = \frac{4p^2}{(\cos\theta + p^2 - q^2)^2 + 4p^2q^2}. \tag{61}$$

For $0 < p, q < 1$ with $p^2 + q^2 = 1$, $\ell(\theta)$ is positive and finite for $0 \le \theta \le \pi$. Hence, each horizon at $\mathrm{P}_\pm$ is compact and homeomorphic to S^2. Further, since $\ell(0) = 1$, it is smooth for $0 \le \theta < \pi$. However, since $\ell(\pi) = p^2/q^2$, it has a conic singularity at $\theta = \pi$, i.e., at the point where it contacts with the segment I, except for the case $p = q = 1/\sqrt{2}$, as illustrated in Fig. 10. The deficit angle is the same as that at the open segment I discussed in III A 3; it is positive for $q^2 > p^2$ and negative for $q^2 < p^2$. In particular, in the limit $q \to 0$, the deficit angle tends negative infinity, which is consistent with the fact that each regular portion of the horizon is a half sphere for ZVW2.

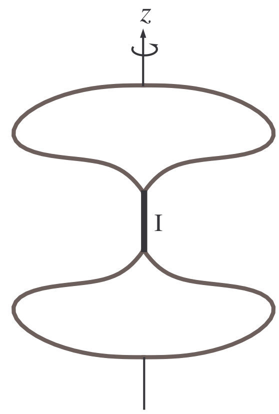

FIG. 10: The shape of a spatial section of the horizon for TS2 with $p = 0.6$.

IV. SUMMARY AND DISCUSSIONS

We have investigated the structure of the domains of the outer communication and their boundaries for the Zipoy-Voorhees-Weyl solution with arbitrary δ and for the Tomimatsu-Sato solution with $\delta = 2$ and $0 < p, q < 1$(TS2). We have shown that three types of singularities arise for these spacetimes. One is the naked curvature singularity. It appears on the segment I: $\rho = 0, -\sigma < z < \sigma$ for the ZVW spacetime with $\delta \neq 0, 1$. We have shown that this singularity is point-like for $\delta < 0$, string-like for $0 < \delta < 1$ and ring-like for $1 < \delta$. The last feature has not been noticed so far, and has a close relation with the existence of the ring singularity for TS2. Another is an inextendible quasi-regular singularity at the segment I for TS2. We have confirmed that this singularity is conical locally, but has a more complicated helical structure globally, as was pointed out by Gibbons and Russell-Clark[4]. The third is the directional quasi-regular singularities at the two points $\mathrm{P}_\pm$: $(\rho, z) = (0, \pm\sigma)$ for the ZVW spacetime with $\delta \geq 2$ and for TS2. We have proved that these points are degenerate Killing horizons for the ZVW spacetime with $\delta = 2, 3$ or $\delta \geq 4$ and for TS2, by constructing regular extensions across these points explicitly.

An interesting feature of the horizon for the ZVW spacetime is that the two regular horizons at $\mathrm{P}_\pm$ share the ring singularity at I as the common boundary and these pieces altogether form a single spherical horizon with a ring singularity at the equator. In contrast, although the two horizons of TS2 also contact with the conical segment I, each of them is a compact surface which is homeomorphic to S^2 and has a conical singularity at the point where it contacts I for $p \neq q$. Thus, it is natural to regard that the horizon of TS2 has two disconnected components. This result is rather surprising because TS2 can be obtained from the Kramer-Neugebauer solution representing a superposition of two Kerr solutions[28], as the limit when the centers of two black holes coincide [29]. This may indicates a new possibility for the final states of gravitational collapse, if the cosmic censorship does not hold.

In this connection, it should be noted that the horizon of the ZVW solution with $\delta = 2$(ZVW2) practically has a single component, as pointed above, although this solution can be also obtained as a similar limit from the Israel-Khan solution[27] with two black holes with the same mass. Further, we have pointed out that the naked curvature singularity of the ZVW spacetime can be regarded to have a positive gravitational mass. This should be contrasted with the fact that the ring singularity of TS2 has zero mass.

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APPENDIX A: GEODESIC EQUATION IN THE ZVW SPACETIME

Geodesics in the ZVW spacetime are determined as solutions to the following set of equations:

$$\dot{t} = E\left(\frac{x+1}{x-1}\right)^{\delta}, \tag{A1a}$$

$$\dot{\phi} = L\frac{(x-1)^{\delta-1}}{(x+1)^{\delta+1}(1-y^2)}, \tag{A1b}$$

$$\frac{\dot{x}^2}{x^2-1} + \frac{\dot{y}^2}{1-y^2} = \frac{(x^2-y^2)^{\delta^2-1}}{(x^2-1)^{\delta^2}}F, \tag{A1c}$$

$$\begin{aligned} F\frac{d^2y}{dx^2} &= F\frac{(\delta^2-1)y(1-y^2)}{(x^2-1)(x^2-y^2)} - L^2\frac{y(x-1)^{2\delta-2}}{(1-y^2)(x+1)^{2\delta+2}} \\ &\quad + \left[\epsilon\frac{(x-1)^{\delta-1}}{(x+1)^{\delta+1}} - F\frac{2(x-\delta)(x^2-y^2)+(\delta^2-1)x(1-y^2)}{(x^2-1)(x^2-y^2)}\right. \\ &\quad \left. -L^2\frac{(x-2\delta)(x-1)^{2\delta-2}}{(1-y^2)(x+1)^{2\delta+2}}\right]\frac{dy}{dx} \\ &\quad - \left[F\frac{y\{x^2-y^2+(1-\delta^2)(1-y^2)\}}{(1-y^2)(x^2-y^2)} + L^2\frac{y(x-1)^{2\delta-1}}{(1-y^2)^2(x+1)^{2\delta+1}}\right]\left(\frac{dy}{dx}\right)^2 \\ &\quad + \left[\frac{\delta\epsilon}{1-y^2}\left(\frac{x-1}{x+1}\right)^{\delta} - F\frac{x(x^2-1)+\delta^2x(1-y^2)-2\delta(x^2-y^2)}{(1-y^2)(x^2-y^2)}\right. \\ &\quad \left. -L^2\frac{(x-2\delta)(x-1)^{2\delta-1}}{(1-y^2)^2(x+1)^{2\delta+1}}\right]\left(\frac{dy}{dx}\right)^3, \end{aligned} \tag{A1d}$$

where the dot denotes the derivative with respect to an affine parameter, and F is given by

$$F := E^2 - L^2\frac{(x-1)^{2\delta-1}}{(1-y^2)(x+1)^{2\delta+1}} - \epsilon\left(\frac{x-1}{x+1}\right)^{\delta}. \tag{A2}$$

E and L are the specific energy and the specific angular momentum of a particle, and ϵ is equal to 1 for time-like geodesics and 0 for null geodesics.

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